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THE NONSTANDARD HULL OF A NORMED SPACE IN A BOOLEAN-VALUED UNIVERSE

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Abstract

In this article, we extend some results of infinitesimal analysis on normed spaces and the field of reals to the functional representation of a Boolean-valued universe. In particular, we prove equivalence of the following three conditions for an arbitrary polyverse over Q: a point $q \in Q$ is not σ-isolated; the stalk of the polyverse at q is countably saturated; and the nonstandard hull of every normed space in the stalk of the polyverse at q is complete.

Key words and phrases: Boolean-valued analysis, infinitesimal analysis, nonstandard analysis, polyverse, nonstandard hull.

One of the most important notions arising in infinitesimal analysis in connection with the theory of normed spaces is that of the nonstandard hull of a normed space, i.e. the set of limited elements factored by the relation of infinite proximity (see, for instance, [2]).

In [1], a convenient functional analog was suggested for a Boolean-valued universe, namely, the set of sections of a corresponding continuous polyverse. In particular, such a functional model makes it possible to consider the notion of infinite proximity of elements of a normed space inside a stalk of the polyverse and thus open an extra opportunity for a synthesis of the methods of infinitesimal and Boolean-valued analysis.

In this connection, the natural question arises of extending the main results of classical infinitesimal analysis to a Boolean-valued universe. In the framework of the problems mentioned, the question of completeness of the nonstandard hull of a normed space inside a stalk of a polyverse is deemed of importance.

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In this article, we give a positive answer to this question in the case when the point of the compact space which corresponds to the stalk is not σ-isolated. We establish completeness of the nonstandard hull using a criterion proven also of countable saturation for a stalk of the universe.

1. Prerequisites

In this section, we give some definitions and facts necessary in the sequel.

Denote by $\mathbb{V}$ the class of all sets and assume $\mathbb{V}$ to be a model of ZFC (more exactly, NBG; see, for instance, [3, II.1.3]).

Throughout, Q is an extremally disconnected compact space.

Denote by $\operatorname{Clop}(Q)$ the totality of all clopen subsets of Q and by $\operatorname{Clop}(q)$, the set of all clopen subsets of Q containing a point $q \in Q$. Let $V^Q \subset Q \times \mathbb{V}$ be a class-correspondence on which a topology (see [1, 1.2]) is defined. The class of all clopen sets of V^Q is denoted by $\operatorname{Clop}(V^Q)$. For every point $q \in Q$, the class $V^Q \cap (\{q\} \times \mathbb{V}) = \{(q,x) \mid (q,x) \in V^Q\}$ is denoted by V^q. The correspondence V^Q is called a *continuous bundle* over Q, and the class V^q is called the *stalk* of V^Q at the point q.

A function $u\colon D \to V^Q$ is called a *section* of V^Q on a set $D \subset Q$ if $u(q) \in V^q$ for all $q \in D$. By a *continuous section* of V^Q we mean a section that is a continuous function. For every subset $D \subset Q$, we denote by $C(D, V^Q)$ the class of all continuous sections of V^Q on D.

As is proven in [1] (Propositions 2.3 and 2.5), if the conditions

(1) $\forall q \in Q \quad \forall x \in V^q \quad \exists u \in C(Q, V^Q) \quad u(q) = x$;

(2) $\forall u \in C(Q, V^Q) \quad \forall A \in \operatorname{Clop}(Q) \quad u(A) \in \operatorname{Clop}(V^Q)$

are satisfied, then V^Q possesses the following properties:

(i) the topology of V^Q is Hausdorff;

(ii) for every $u \in C(Q, V^Q)$ and $q \in Q$, the sets $u(A)$ $\big(A \in \operatorname{Clop}(q)\big)$ form a neighborhood base of the point $u(q)$;

(iii) all elements of $C(Q, V^Q)$ are open and closed mappings;

(iv) the topology of V^Q is extremally disconnected.

In what follows, we assume that, for every point $q \in Q$, the class V^q is an algebraic system of signature $\{\in\}$.

For a formula $\varphi(t_1, \dots, t_n)$ of set-theoretic signature and sections $u_1, \dots, u_n$ of the bundle V^Q, denote by $\{\varphi(u_1, \dots, u_n)\}$ the set

$$\big\{q \in \operatorname{dom} u_1 \cap \dots \cap \operatorname{dom} u_n \bigm| V^q \models \varphi\big(u_1(q), \dots, u_n(q)\big)\big\}.$$

Given an $x \in \mathbb{U}$, where $\mathbb{U}$ is an algebraic system of signature $\{\in\}$, the class $x{\downarrow} := \{y \in \mathbb{U} \mid \mathbb{U} \models y \in x\}$ is called the *descent* of x. If $\mathbb{U}$ meets the extensionality axiom then the equalities $x{\downarrow} = y{\downarrow}$ and $x = y$ are equivalent for every $x, y \in \mathbb{U}$. Below, we will be interested mainly in the case of $\mathbb{U} = V^q$.

For an arbitrary section $u \in C(Q, V^Q)$, the class $\bigcup_{q\in Q} u(q)\!\downarrow$ is called the *unpack* of the section u and denoted by $\llcorner u \lrcorner$.

A continuous bundle V^Q is called a (*continuous*) *polyverse* over Q if the extensionality and regularity axioms are true in every stalk V^q $(q \in Q)$ and, in addition to (1) and (2), the following conditions hold:

(3) $\forall\, u \in C(Q, V^Q) \quad \llcorner u \lrcorner \in \mathrm{Clop}(V^Q)$;

(4) $\forall\, X \in \mathrm{Clop}(V^Q) \quad \exists\, u \in C(Q, V^Q) \quad \llcorner u \lrcorner = X$.

For arbitrary sections $u, v \in C(Q, V^Q)$, the sets $\{u = v\}$ and $\{u \in v\}$ are clopen (see [1, 3.3]), which allows us to introduce two class-functions

$$\|\cdot = \cdot\|,\ \|\cdot \in \cdot\| : C(Q, V^Q) \times C(Q, V^Q) \to \mathrm{Clop}(Q)$$

by setting $\|u = v\| = \{u = v\}$ and $\|u \in v\| = \{u \in v\}$.

It is easy to verify that the triple $\big(C(Q, V^Q),\ \|\cdot = \cdot\|,\ \|\cdot \in \cdot\|\big)$ is a separated $\mathrm{Clop}(Q)$-valued algebraic system.

As demonstrated by the following theorem, the class of continuous sections of a polyverse is the general form of a Boolean-valued universe.

Theorem [1, 4.10]. *Let Q be the Stone space of a complete Boolean algebra B.*

(a) *The class $C(Q, V^Q)$ of continuous sections of a polyverse V^Q over Q is a Boolean-valued universe over* $\mathrm{Clop}(Q)$.

(b) *For an arbitrary Boolean-valued universe $\mathfrak{U}$ over B there is a polyverse V^Q over Q such that $C(Q, V^Q)$ is isomorphic to $\mathfrak{U}$.*

More detailed information about continuous bundles and a continuous polyverse can be found in [1].

Throughout the article, $\mathbb{V}^Q$ is a continuous polyverse and $\mathbb{V}^q$ is its stalk at a point $q \in Q$ which we will also denote by ${}^q\mathbb{V}$. (Note that ${}^q\mathbb{V}$ is a model of ZFC.)

Let us agree to denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of reals and naturals $(0 \notin \mathbb{N})$, and by $\mathcal{R}$ and $\mathcal{N}$, the elements of $C(Q, \mathbb{V}^Q)$ which are the sets of reals and naturals in $C(Q, \mathbb{V}^Q)$. Recall that $\mathcal{N} = \mathbb{N}^\wedge$, where $(\cdot)^\wedge$ is the canonical embedding of $\mathbb{V}$ into $C(Q, \mathbb{V}^Q)$ (see [3, II.2.2.7]). Introduce also the notations ${}^q\mathbb{R} = \mathcal{R}(q)$, ${}^q\mathbb{N} = \mathcal{N}(q)$; and if $\alpha \in \mathbb{R}$ then we denote by ${}^q\alpha$ the element $\alpha^\wedge(q) \in {}^q\mathbb{V}$.

If an element $X \in {}^q\mathbb{V}$ is a field or an ordered set inside ${}^q\mathbb{V}$ then $X\!\downarrow$ is naturally endowed by field operations or an order relation respectively. For example, for $\alpha, \beta \in {}^q\mathbb{R}\!\downarrow$, the sum $\alpha + \beta$ is defined as the element $\gamma \in {}^q\mathbb{R}\!\downarrow$ such that ${}^q\mathbb{V} \models (\gamma = \alpha + \beta)$. It is easy to check that the descent of a field is a field. If $X \in {}^q\mathbb{V}$ is a vector space inside ${}^q\mathbb{V}$ over a field $F \in {}^q\mathbb{V}$ then $X\!\downarrow$ is a vector space over $F\!\downarrow$.

Lemma 1. *For every real $\alpha \in \mathbb{R}$, ${}^q\alpha \in {}^q\mathbb{V}$ is a real inside ${}^q\mathbb{V}$, i.e., ${}^q\alpha \in {}^q\mathbb{R}{\downarrow}$. The function ${}^q(\cdot)\colon \mathbb{R} \to {}^q\mathbb{R}{\downarrow}$ is injective and preserves the order relation and the operations of addition and multiplication. Furthermore, $\mathbb{N}^{\wedge}(q) = {}^q\mathbb{N}$ and $\mathbb{R}^{\wedge}(q) \subset {}^q\mathbb{R}$ inside ${}^q\mathbb{V}$.*

It is easy to prove this lemma using the properties of the canonical embedding $(\cdot)^{\wedge}$ cited in [3] and basing on the classical definition of number sets in which naturals are finite ordinals, rationals are equivalence classes of pairs of naturals, and reals are defined as Dedekind sections of the set of rationals.

In what follows, we identify $\alpha \in \mathbb{R}$ and ${}^q\alpha \in {}^q\mathbb{R}{\downarrow}$ and thus assume that $\mathbb{R} \subset {}^q\mathbb{R}{\downarrow}$.

Let $\mathbb{U}$ be a model of set-theoretic signature and let $P \in \mathbb{U}$ be a relation (i.e. a set of pairs) inside $\mathbb{U}$. Consider the following property of a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $(\operatorname{dom} P){\downarrow}$: if, for every $n \in \mathbb{N}$, there is an element $y_n \in \mathbb{U}$ such that

$$\mathbb{U} \models \big((x_1, y_n), \dots, (x_n, y_n) \in P\big)$$

then there is a $y \in \mathbb{U}$ for which

$$\mathbb{U} \models \big((x_n, y) \in P\big) \quad \text{for all } n \in \mathbb{N}.$$

The relation P is called *countably saturated* if every sequence of elements in $(\operatorname{dom} P){\downarrow}$ possesses the above property. The model $\mathbb{U}$ is said to be *countably saturated* if every relation in $\mathbb{U}$ is countably saturated.

A point in a topological space is called *σ-isolated* (or a *P-point*) if the intersection of every countable set of its neighborhoods is a neighborhood of the point.

2. Nonstandard hull of a normed space

In this section, we establish a criterion of countable saturation for a stalk of a polyverse, suggest analogs to some theorems of infinitesimal analysis on the field of reals and normed spaces, and study the problem of completeness of the nonstandard hull of a normed space in a stalk of the polyverse.

Theorem 1. *The class ${}^q\mathbb{V}$ is countably saturated if and only if the point $q \in Q$ is not σ-isolated.*

$\lhd$ Sufficiency. Suppose that q is not σ-isolated, and prove countable saturation of ${}^q\mathbb{V}$.

Let P be an element of ${}^q\mathbb{V}$ that is a relation inside ${}^q\mathbb{V}$. Consider an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $(\operatorname{dom} P){\downarrow}$ and assume that, for every $n \in \mathbb{N}$, there is a $y_n \in {}^q\mathbb{V}$ meeting the condition

$${}^q\mathbb{V} \models \big((x_1, y_n), \dots, (x_n, y_n) \in P\big).$$

We demonstrate that there is a $y \in {}^q\mathbb{V}$ with ${}^q\mathbb{V} \models \big((x_n, y) \in P\big)$ for all $n \in \mathbb{N}$.

Draw a section $\mathcal{P} \in C(Q, \mathbb{V}^Q)$ through P and sections $u_n, v_n \in C(Q, \mathbb{V}^Q)$ $(n \in \mathbb{N})$ through x_n and y_n. Define the sets $W_n \in \mathrm{Clop}(Q)$ $(n \in \mathbb{N})$ as follows:

$$\begin{aligned}
&W_1 = \|(u_1, v_1) \in \mathcal{P}\|,\\
&W_2 = \|(u_1, v_2), (u_2, v_2) \in \mathcal{P}\| \cap W_1,\\
&\dots\\
&W_n = \|(u_1, v_n), \dots, (u_n, v_n) \in \mathcal{P}\| \cap W_{n-1},\\
&\dots
\end{aligned}$$

Case 1: $\bigcap_{n\in\mathbb{N}} W_n$ is not a neighborhood of q.

Put $D_1 = W_1 \backslash W_2, \dots, D_n = W_n \backslash W_{n+1}, \dots$ and denote by D_0 the complement of the closure of $\bigcup_{n\in\mathbb{N}} D_n$ to Q. The sets D_n $(n = 0, 1, 2, \dots)$ form a partition of unity in the Boolean algebra $\mathrm{Clop}(Q)$.

Define a section $\tilde{v} \in C\big(\bigcup_{n=0}^{\infty} D_n, \mathbb{V}^Q\big)$ by setting $\tilde{v}|_{D_n} = v_n|_{D_n}$ for all $n \in \mathbb{N}$ and $\tilde{v}|_{D_0} = \varnothing^{\wedge}|_{D_0}$. By the extension principle (see [1, 3.8]), the section $\tilde{v}$ (defined on a dense subset of Q) is extendable to a section $v \in C(Q, \mathbb{V}^Q)$. We will demonstrate that the element $y = v(q)$ is a sought one. To prove that, fix a number $n \in \mathbb{N}$ and validate the containment

$$q \in \|(u_n, v) \in \mathcal{P}\|.$$

Put $W_0 = \|(u_n, v) \in \mathcal{P}\| \cap W_n$ and prove that q belongs to the closure of W_0 (and hence $q \in W_0$). Suppose to the contrary that some neighborhood $W \subset W_n$ of q does not intersect W_0. As is easily observed, $D_m \subset W_0$ for $m \geqslant n$. Consequently, $W \subset W_n \backslash \bigcup_{m\geqslant n} D_m$. On the other hand, the set

$$W_n \backslash \bigcup_{m\geqslant n} D_m = \bigcap_{m\geqslant n} W_m = \bigcap_{m\in\mathbb{N}} W_m$$

is not a neighborhood of q.

Case 2: $\bigcap_{n\in\mathbb{N}} W_n$ is a neighborhood of q.

Since q is not σ-isolated, it follows that there exists a decreasing sequence of clopen neighborhoods $(W_n')_{n\in\mathbb{N}}$ of q such that $\bigcap_{n\in\mathbb{N}} W_n'$ is not a neighborhood of q. Without loss of generality we may assume that $W_n' \subset \bigcap_{m\in\mathbb{N}} W_m$ for all $n \in \mathbb{N}$. The rest of the proof amounts to repeating the argument for Case 1 with substituting W_n' for W_n.

Necessity. Assume that q is a σ-isolated point. We prove that ${}^q\mathbb{V}$ is not countably saturated.

Let P be an element of ${}^q\mathbb{V}$ that is equal inside ${}^q\mathbb{V}$ to the set of all pairs (x, y) of elements of ${}^q\mathbb{N}$ such that $x \neq y$. We prove that the relation P is not countably saturated.

Obviously, ${}^q\mathbb{V} \models \Big(({}^q1, {}^q(n+1)), \dots, ({}^qn, {}^q(n+1)) \in P\Big)$ for all $n \in \mathbb{N}$. Suppose to the contrary that there exists a $y \in {}^q\mathbb{N}{\downarrow}$ meeting the relation ${}^qn \neq y$ for all $n \in \mathbb{N}$. Draw a section $v \in C(Q, \mathbb{V}^Q)$ through y. Then

$$q \in \|v \in \mathbb{N}^\wedge\| = \bigvee_{n \in \mathbb{N}} \|v = n^\wedge\| = \mathrm{cl} \bigcup_{n \in \mathbb{N}} \|v = n^\wedge\|.$$

Since q does not belong to $\bigcup_{n \leqslant m} \|v = n^\wedge\|$ for any $m \in \mathbb{N}$, it follows that $q \in \mathrm{cl} \bigcup_{n > m} \|v = n^\wedge\| =: W_m$ for all $m \in \mathbb{N}$, i.e., $q \in \bigcap_{m \in \mathbb{N}} W_m$. On the other hand, using pairwise disjointness of $\|v = n^\wedge\|$ $(n \in \mathbb{N})$, it is easy to prove that the intersection $\bigcap_{m \in \mathbb{N}} W_m$ is not a neighborhood of q and thus get a contradiction to the fact that q is σ-isolated. $\rhd$

Proposition 1. *For every set $C \subset {}^q\mathbb{V}$, there exists a $\widetilde{C} \in {}^q\mathbb{V}$ such that $C \subset \widetilde{C}{\downarrow}$.*

$\lhd$ Through every $c \in C$, draw a section $u_c \in C(Q, \mathbb{V}^Q)$ and put $u = \{u_c \mid c \in C\}{\uparrow}$. Clearly, $\widetilde{C} = u(q)$ is a sought element. $\rhd$

Proposition 2. *For every finite set $C \subset {}^q\mathbb{V}$, there exists a $\overline{C} \in {}^q\mathbb{V}$ such that $C = \overline{C}{\downarrow}$. Moreover, $\overline{C}$ is a finite set inside ${}^q\mathbb{V}$.*

$\lhd$ Suppose that $C = \{x_1, \dots, x_n\} \subset {}^q\mathbb{V}$. Denote by u the ascent of the set of sections $\{u_1, \dots u_n\} \subset C(Q, \mathbb{V}^Q)$ drawn through $x_1, \dots, x_n$ and put $\overline{C} = u(q)$. Obviously, $x_1, \dots, x_n \in \overline{C}{\downarrow}$. We demonstrate that every element $x \in \overline{C}{\downarrow}$ coincides with x_i for some $i \in \{1, \dots, n\}$. Draw a section $v \in C(Q, \mathbb{V}^Q)$ through x. Obviously,

$$q \in \|v \in u\| = \bigvee_{i=1}^{n} \|v = u_i\| = \bigcup_{i=1}^{n} \|v = u_i\|$$

and hence $x = v(q) = u_i(q) = x_i$ for some $i \in \{1, \dots, n\}$. $\rhd$

Clearly, the element $\overline{C}$ of Proposition 2 is uniquely defined by the finite set C. We will call this element the *ascent* of C and denote it by $C{\uparrow}$.

Suppose that $C \subset {}^q\mathbb{V}$, $D \in {}^q\mathbb{V}$, and $f \colon C \to D{\downarrow}$. An element $\overline{f} \in {}^q\mathbb{V}$ is called an *internal extension* of f if ${}^q\mathbb{V} \models \overline{f} \colon \overline{C} \to D$, where $\overline{C} \in {}^q\mathbb{V}$, $C \subset \overline{C}{\downarrow}$, and ${}^q\mathbb{V} \models \overline{f}(c) = f(c)$ for all $c \in C$.

Lemma 2. *Suppose that $q \in Q$ is not σ-isolated. If C is a countable subset of ${}^q\mathbb{V}$ and $D \in {}^q\mathbb{V}$ then every function $f \colon C \to D{\downarrow}$ has an internal extension.*

$\lhd$ By Proposition 1, we can find a $\widetilde{C} \in {}^q\mathbb{V}$ with $C \subset \widetilde{C}{\downarrow}$. Assume that, inside ${}^q\mathbb{V}$, $P \in {}^q\mathbb{V}$ is a relation on the set of all functions from subsets of $\widetilde{C}$ into D defined as follows:

$$(h, \overline{h}) \in P \;\Leftrightarrow\; \overline{h} \text{ is an extension of } h.$$

Let $C = \{c_1, \dots, c_n, \dots\}$. For every $n \in \mathbb{N}$, denote by f_n an element of ${}^q\mathbb{V}$ such that ${}^q\mathbb{V} \models f_n = \{(c_n, f(c_n))\}$ and put

$$\bar{f}_n = \{(c_1, f(c_1)), \dots, (c_n, f(c_n))\}\!\uparrow.$$

Clearly, ${}^q\mathbb{V} \models ((f_1, \bar{f}_n), \dots, (f_n, \bar{f}_n) \in P)$. Then the countable saturation property of ${}^q\mathbb{V}$ (guaranteed by Theorem 1) implies existence of an $\bar{f} \in {}^q\mathbb{V}$ such that ${}^q\mathbb{V} \models ((f_n, \bar{f}) \in P)$ for all $n \in \mathbb{N}$. Obviously, $\bar{f}$ is a sought internal extension of f. $\triangleright$

We call elements of ${}^q\mathbb{R}\!\downarrow$ *internal numbers.* An internal number λ is said to be *standard* if there exists a number $\alpha \in \mathbb{R}$ such that ${}^q\alpha = \lambda$. We observe that, reckoning with the above agreement (on the inclusion $\mathbb{R} \subset {}^q\mathbb{R}\!\downarrow$), we identify standard numbers and elements of $\mathbb{R}$.

We call an internal number whose modulus is less than some standard number a *limited number* and denote by $\mathcal{O}({}^q\mathbb{R})$ the set of all limited numbers. Numbers that are not limited are called *unlimited.*

An internal number λ is said to be *infinitesimal* if $|\lambda| < \alpha$ for every standard number $\alpha > 0$. We say that two internal numbers are *infinitely close* if their difference is infinitesimal. We denote by $\approx$ the corresponding relation of infinite proximity. This is an equivalence relation on the set of internal numbers. Denote by $[\lambda]$ the equivalence class of an internal number λ.

Proposition 3. *For every limited number λ, there exists a unique standard number infinitely close to λ.*

$\triangleleft$ Existence. Consider the sets

$$A = \{\alpha \in \mathbb{R} \mid \alpha < \lambda\}, \quad B = \{\beta \in \mathbb{R} \mid \beta > \lambda\}.$$

Limitedness of λ implies that $A, B \neq \varnothing$. Moreover, $A \cup B = \mathbb{R}$ and $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$. Put $\gamma = \sup A = \inf B$. If $\gamma < \lambda$ then $\gamma \in A$ and hence $\gamma \approx \lambda$ (because, in this case, γ is the maximal standard number less than λ). The case $\gamma > \lambda$ can be considered in the same way.

Uniqueness. If standard numbers α and β are such that $\alpha \neq \beta$ and $\alpha, \beta \approx \lambda$ then the number $|\alpha - \beta|$ is nonzero and infinitesimal, which contradicts its standardness. $\triangleright$

Corollary. *The mapping $\alpha \mapsto [\alpha]$ is a bijection from $\mathbb{R}$ onto $\mathcal{O}({}^q\mathbb{R})/\!\approx$.*

For every limited number λ, the standard number infinitely close to it is called the *standard part* of λ and denoted by ${}^\circ\lambda$.

Proposition 4. *Suppose that a point $q \in Q$ is not σ-isolated. Assume that the descent of an element $D \in {}^q\mathbb{V}$ includes the set $\{n \in \mathbb{N} \mid m \leqslant n\}$ for some $m \in \mathbb{N}$. Then there exists an unlimited natural M such that the inclusion*

$$\{n \in {}^q\mathbb{N} \mid m \leqslant n < M\} \subset D$$

holds inside ${}^q\mathbb{V}$.

$\lhd$ It suffices to consider the relation P defined inside ${}^q\mathbb{V}$ by the rule

$$(x, y) \in P \Leftrightarrow \big(x, y \in D \cap {}^q\mathbb{N},\ x < y \text{ and } \{n \in {}^q\mathbb{N} \mid x \leqslant n < y\} \subset D\big)$$

and use the countable saturation property of ${}^q\mathbb{V}$. $\rhd$

Suppose that $X \in {}^q\mathbb{V}$ is a normed space over ${}^q\mathbb{R}$ inside ${}^q\mathbb{V}$. We call an $x \in X{\downarrow}$ *limited* if its norm inside ${}^q\mathbb{V}$ is a limited number. Denote by $\mathcal{O}(X)$ the set of all limited elements in $X{\downarrow}$.

We say that $x, y \in X{\downarrow}$ are *infinitely close* and write $x \approx y$ if the norm of their difference inside ${}^q\mathbb{V}$ is infinitesimal. The corresponding relation of infinite proximity is an equivalence relation on $X{\downarrow}$. We denote the equivalence class of $x \in X{\downarrow}$ by $[x]$.

Denote $\mathcal{O}(X)/{\approx}$ by $\mathbf{X}$ and put

$$\mathbf{x} + \mathbf{y} = [x + y], \quad \lambda\mathbf{x} = [\lambda x], \quad \|\mathbf{x}\| = {}^\circ\|x\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $\lambda \in \mathbb{R}$, where x and y are arbitrary representatives of the classes $\mathbf{x}$ and $\mathbf{y}$ and $\|x\|$ is the norm of x inside ${}^q\mathbb{V}$. It is easy to prove that these operations are well defined and turn $\mathbf{X}$ into a normed space over $\mathbb{R}$ which we denote by $\widehat{X}$ and call the *nonstandard hull* of the normed space X.

Theorem 2. *Suppose that a point $q \in Q$ is not σ-isolated and $X \in {}^q\mathbb{V}$ is a normed space inside ${}^q\mathbb{V}$. Then the normed space $\widehat{X}$ is complete.*

$\lhd$ Assume that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\widehat{X}$ and $x_n \in \mathbf{x}_n$ for all $n \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, there exists a number $n_k \in \mathbb{N}$ such that ${}^q\mathbb{V} \models \|x_n - x_m\| < 1/k$ for all $n, m > n_k$. Consider an internal extension $(x_n)_{n\in\widetilde{N}} \in {}^q\mathbb{V}$ of the sequence $(x_n)_{n\in\mathbb{N}}$ regarded as a function from $\mathbb{N}$ into $X{\downarrow}$ (such an extension exists by Lemma 2). We may assume that ${}^q\mathbb{V} \models \widetilde{N} \subset {}^q\mathbb{N}$.

We demonstrate that there exists an unlimited number $\overline{m} \in {}^q\mathbb{N}{\downarrow}$ belonging to $\widetilde{N}{\downarrow}$ and satisfying inside ${}^q\mathbb{V}$ the inequality $\|x_n - x_{\overline{m}}\| < 1/k$ for all $k, n \in \mathbb{N}$, $n > n_k$. (We thus prove that $x_{\overline{m}} \in X{\downarrow}$ is limited and $(\mathbf{x}_n)_{n\in\mathbb{N}}$ converges to $[x_{\overline{m}}]$.)

For every $k \in \mathbb{N}$, denote by E_k the element of ${}^q\mathbb{V}$ meeting inside ${}^q\mathbb{V}$ the relation

$$E_k = \big\{m \in \widetilde{N} \mid \|x_i - x_j\| < 1/k \text{ for all } i, j \in \widetilde{N} \text{ such that } n_k < i, j \leqslant m\big\}.$$

Note that $\mathbb{N} \subset E_k{\downarrow}$ and hence, by Proposition 4, we have the inclusion

$$\{m \in {}^q\mathbb{N}{\downarrow} \mid m \leqslant m_k\} \subset E_k{\downarrow}$$

for some unlimited number $m_k \in \widetilde{N}{\downarrow}$. We may suppose that $m_{k+1} < m_k$ for all $k \in \mathbb{N}$. Since ${}^q\mathbb{V}$ is countably saturated, there exists an unlimited number $m \in \widetilde{N}{\downarrow}$ such that $m < m_k$ for all $k \in \mathbb{N}$. Then $m \in E_k{\downarrow}$ for every $k \in \mathbb{N}$, whence $\overline{m} = m$ is a sought internal number. $\rhd$

Proposition 5. *If a point $q \in Q$ is σ-isolated then all reals inside ${}^q\mathbb{V}$ are standard.*

$\vartriangleleft$ We first prove that ${}^q\mathbb{V}$ has no nonstandard naturals. Suppose to the contrary that there exists a section $u \in C(Q, \mathbb{V}^Q)$ for which $C(Q, \mathbb{V}^Q) \models u \in \mathcal{N}$ and $u(q) \neq {}^q k$ for all $k \in \mathbb{N}$. Since q is σ-isolated, it follows that $W = \bigcap_{k\in\mathbb{N}} \|u \neq k^\wedge\|$ is a neighborhood of q. On the other hand,

$$Q = \|u \in \mathcal{N}\| = \bigvee_{k\in\mathbb{N}} \|u = k^\wedge\| = \mathrm{cl} \bigcup_{k\in\mathbb{N}} \|u = k^\wedge\|;$$

therefore, $\bigcup_{k\in\mathbb{N}} \|u = k^\wedge\| = Q \setminus W$ is dense in Q, which is impossible.

This implies that ${}^q\mathbb{V}$ has no unlimited reals. To complete the proof, it suffices to note that if $\lambda \in {}^q\mathbb{R}{\downarrow}$ is a nonstandard limited number then $1/(\lambda - {}^\circ\lambda)$ is unlimited. $\vartriangleright$

Proposition 5 means that, in the case of a σ-isolated point q, the equalities ${}^q\mathbb{R}{\downarrow} = \mathbb{R}$ and ${}^q\mathbb{N}{\downarrow} = \mathbb{N}$ hold (with the above agreement on the inclusion $\mathbb{R} \subset {}^q\mathbb{R}{\downarrow}$ taken into account).

We now prove the converse to Theorem 2.

Proposition 6. *Let $q \in Q$ be a σ-isolated point. Then there exists an element $X \in {}^q\mathbb{V}$ that is a normed space inside ${}^q\mathbb{V}$ with $\widehat{X}$ incomplete.*

$\vartriangleleft$ Consider the element $X \in {}^q\mathbb{V}$ which is an incomplete normed space inside ${}^q\mathbb{V}$ and prove that its nonstandard hull $\widehat{X}$ is incomplete. From Proposition 5 it follows that infinite proximity of elements of $X{\downarrow}$ is equivalent to their coincidence, and hence the elements of $\widehat{X}$ are singletons $\{x\}$ of limited elements $x \in X{\downarrow}$.

Let s be an element of ${}^q\mathbb{V}$ which is a Cauchy sequence of elements of X without a limit. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $X{\downarrow}$ by setting x_n equal to the ${}^q n$th term of s inside ${}^q\mathbb{V}$. Making use of Proposition 5, it is easy to see that $(\{x_n\})_{n\in\mathbb{N}}$ is a Cauchy sequence of elements of $\widehat{X}$. On the other hand, this sequence cannot have a limit because its convergence to an $\{x\} \in \widehat{X}$ would obviously imply convergence of s to x inside ${}^q\mathbb{V}$. $\vartriangleright$

In conclusion, we formulate a theorem that summarizes the main results of this article.

Theorem 3. *Let ${}^q\mathbb{V}$ be a stalk at a point $q \in Q$ of a continuous polyverse over an extremally disconnected compact space Q. The following assertions are equivalent:*

(a) *q is not σ-isolated;*
(b) *${}^q\mathbb{V}$ is countably saturated;*
(c) *the nonstandard hull of every normed space inside ${}^q\mathbb{V}$ is complete.*

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