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Discrimination Between B-Processes is Impossible

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Abstract Two series of binary observations $x_1, x_1, \dots$ and $y_1, y_2, \dots$ are presented: x_n and y_n are given at each time $n \in \mathbb{N}$. It is assumed that the sequences are generated independently of each other by two B-processes. The question of interest is whether the sequences represent a typical realization of two different processes or of the same one. It is demonstrated that this is impossible to decide, in the sense that every discrimination procedure is bound to err with non-negligible frequency when presented with sequences from some B-processes. This contrasts with earlier positive results on B-processes, in particular, those showing that there are consistent $\bar{d}$-distance estimates for this class of processes, and on ergodic processes, in particular, those establishing consistent change point estimates.

Keywords Process discrimination · B-processes · Stationary ergodic processes · Time series · Homogeneity testing

Mathematics Subject Classification (2000) Primary 62G10 · 60G10 · Secondary 62M07 · 37A99 · 60J10

1 Introduction

Two series of binary observations $x_1, x_1, \dots$ and $y_1, y_2, \dots$ are presented sequentially. A *discrimination procedure* D is a family of mappings $D_n : X^n \times X^n \to \{0, 1\}$, $n \in \mathbb{N}$, $X = \{0, 1\}$, that maps a pair of samples $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ into a binary ("yes" or "no") answer: the samples are generated by different distributions, or they are generated by the same distribution.

A discrimination procedure D is *asymptotically correct for a set $\mathcal{C}$ of process distributions* if, for any two distributions $\rho_x, \rho_y \in \mathcal{C}$ independently generating the

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sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$, respectively, the expected output converges to the correct answer: the following limit exists, and the equality holds:

$$\lim_{n\to\infty} \mathbf{E} D_n\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \begin{cases} 0 & \text{if } \rho_x = \rho_y, \\ 1 & \text{otherwise.} \end{cases}$$

This is perhaps the weakest notion of correctness one can consider. Clearly, asymptotically correct discriminating procedures exist for many classes of processes, for example, for the class of all i.i.d. processes (e.g., [4]) and various parametric families.

We show that there is no asymptotically correct discrimination procedure for the class of all B-processes (see the definition below), meaning that for any discrimination, the expected answer does not converge to the correct one for some processes. The class of B-processes is sufficiently wide to include, for example, k-order Markov processes and functions of them, but, on the other hand, it is a strict subset of the set of stationary ergodic processes. B-processes play an important role in such fields as information theory and ergodic theory [7, 15, 16].

Previously, Ornstein and Weiss [9] and Ornstein and Shields [8] showed that consistent estimates of $\bar{d}$-distance (defined below) for B-processes exist, while it is impossible to estimate this distance outside this class. The latter result, as well as the result of the present work, contrasts with the positive results of [6, 11–13], which show, in particular, that asymptotically consistent change-point estimation is possible for stationary ergodic real-valued processes. Thus, we can say that discrimination is harder than distance estimation and change-point estimation. The result of this work also complements earlier negative results on B-processes and stationary ergodic processes, such as [1, 2, 5, 10, 14], that establish negative results concerning prediction, density estimation, testing membership to certain families of processes, and others. The construction used in the proof of the result of this work is somewhat similar to that of [1] used to show that consistent density estimation is impossible for stationary ergodic processes (although the latter uses the method of cutting and stacking rather than Markov chains employed here).

Next, we define the $\bar{d}$ distance and B-processes (mainly following [9] in our formulations) and give more precise formulations of some of the existing results mentioned above. The main result of this work is formulated and proven in the next section.

For two finite-valued stationary processes ρ_x and ρ_y, the *$\bar{d}$-distance* $\bar{d}(\rho_x, \rho_y)$ is said to be less than ε if there exists a single stationary process ν_{xy} on pairs (x_n, y_n), $n \in \mathbb{N}$, such that x_n, $n \in \mathbb{N}$, are distributed according to ρ_x, and y_n are distributed according to ρ_y, while

$$\nu_{xy}(x_1 \neq y_1) \le \varepsilon. \tag{1}$$

The infimum of the ε's for which a coupling can be found such that (1) is satisfied is taken to be the $\bar{d}$-distance between ρ_x and ρ_y.

A process is called a *B-process* (or a Bernoulli process) if it is in the $\bar{d}$-closure of the set of all aperiodic stationary ergodic k-step Markov processes, where $k \in \mathbb{N}$. For more information on $\bar{d}$-distance and B-processes, the reader is referred to [7].

As it was mentioned, [9] constructs an estimator $\bar{s}_n$ such that

$$\lim_{n\to\infty} \bar{s}_n\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = \bar{d}(\rho_1,\rho_2) \quad \rho_1\times\rho_2\text{-a.s.} \tag{2}$$

if both processes ρ_1 and ρ_2 generating the samples x_i and y_i, respectively, are B-processes. In the same work it is shown that there is no estimator $\bar{s}_n$ for which (2) holds for every pair ρ_1, ρ_2 of stationary ergodic processes. Some extensions of these results are given in [8].

It is interesting to compare these results to those that are obtained for a weaker process distance, the distributional distance. (As far as the results of the present work are concerned, this distance is only used in the proof.) It is defined as follows. Denote by X^* the set of all finite tuples $X^* := \bigcup_{k\in\mathbb{N}} X^k$. Assuming length-lexicographical order on X^*, introduce the notation $X^* = \{B_1, B_2, \dots\}$ for the elements of this set, and let $|B_i|$ denote the length of B_i (that is, $|B_i| = k$ if $B_i \in X^k$). Furthermore, define the weights $w_k := 2^{-k}$. For (arbitrary) process distributions ρ_1, ρ_2, the distributional distance $d(\rho_1,\rho_2)$ between them is defined as

$$d(\rho_1,\rho_2) = \sum_{i=1}^{\infty} w_i \big|\rho_1\big((x_1,\dots,x_{|B_i|}) = B_i\big) - \rho_2\big((x_1,\dots,x_{|B_i|}) = B_i\big)\big|. \tag{3}$$

We refer to [3] for more information on distributional distance and its properties. Notably, this distance is weaker than the $\bar{d}$ distance. In [12, 13] an estimator $\bar{s}_n((x_1,\dots,x_n),(y_1,\dots,y_n))$ is constructed such that

$$\lim_{n\to\infty} s_n\big((x_1,\dots,x_n),(y_1,\dots,y_n)\big) = d(\rho_1,\rho_2) \quad \rho_1\times\rho_2\text{-a.s.}$$

if both processes ρ_1 and ρ_2 generating the samples x_i and y_i, respectively, are stationary ergodic. This estimator is also used to construct a consistent change-point estimate. That is, given a sample

$$(z_1,\dots,z_n) = (x_1,\dots,x_{\theta n}, y_{\theta n+1},\dots,y_n)$$

which is a concatenation of two samples generated by two different stationary ergodic processes, with the point of change (concatenation) $\theta \in (0,1)$ being the unknown parameter to estimate, there is an estimator $\hat{\theta}_n$ such that $|\hat{\theta}_n - \theta| = o(1)$ almost surely as the size n of the sample goes to infinity. This holds even for real-valued processes [12]. On the other hand, the results of the present work imply that one cannot consistently tell whether there is a change in the sample or not.

Summarizing, we can say that the stronger the distance, the harder it is to estimate: the distributional distance can be consistently estimated for stationary ergodic processes, the $\bar{d}$ distance can be consistently estimated for B-processes but not for stationary ergodic processes, while, as is shown in this work, the strongest possible distance—the one that gives discrete topology—cannot be consistently estimated for B-processes.

2 The Main Result

The main result of this work is the theorem below. The construction on which the proof is based uses the ideas of the construction of Ryabko [10] to demonstrate that consistent prediction for stationary ergodic processes is impossible (see also the modification of this construction in [2]).

Theorem 1 *There is no asymptotically correct discrimination procedure for the set of all B-processes.*

Proof We will assume that asymptotically correct discrimination procedure D for the class of all B-processes exists and will construct a B-process ρ such that if both sequences x_i and y_i, $i \in \mathbb{N}$, are generated by ρ, then $\mathbf{E}D_n$ diverges; this contradiction will prove the theorem.

The scheme of the proof is as follows. In Step 1 we construct a sequence of processes ρ_{2k}, ρ_{d2k+1}, and ρ_{u2k+1}, where $k = 0, 1, \ldots.$ In Step 2 we construct a process ρ, which is shown to be the limit of the sequence ρ_{2k}, $k \in \mathbb{N}$, in $\bar{d}$-distance. In Step 3 we show that two independent runs of the process ρ have the property that (with high probability) they first behave like two runs of a single process ρ_0, then like two runs of two different processes ρ_{u1} and ρ_{d1}, then like two runs of a single process ρ_2, and so on, thereby showing that the test D diverges and obtaining the desired contradiction.

Assume that there exists an asymptotically correct discriminating procedure D. Fix some $\varepsilon \in (0, 1/2)$ and $\delta \in [1/2, 1)$ to be defined in Step 3.

Step 1. We will construct the sequence of process ρ_{2k}, ρ_{u2k+1}, and ρ_{d2k+1}, where $k = 0, 1, \ldots.$

Step 1.0. Construct the process ρ_0 as follows. A Markov chain m_0 is defined on the set $\mathbb{N}$ of states. From each state $i \in \mathbb{N}$ the chain passes to the state 0 with probability δ and to the state $i + 1$ with probability $1 - \delta$. With transition probabilities so defined, the chain possesses a unique stationary distribution M_0 on the set $\mathbb{N}$, which can be calculated explicitly by using, e.g., [17, Theorem VIII.4.1] and is as follows: $M_0(0) = \delta$ and $M_0(k) = \delta(1 - \delta)^k$ for all $k \in \mathbb{N}$. Take this distribution as the initial distribution over the states.

The function f_0 maps the states to the output alphabet $\{0, 1\}$ as follows: $f_0(i) = 1$ for every $i \in \mathbb{N}$. Let s_t be the state of the chain at time t. The process ρ_0 is defined as $\rho_0 = f_0(s_t)$ for $t \in \mathbb{N}$. As a result of this definition, the process ρ_0 simply outputs 1 with probability 1 on every time step (however, by using different functions f we will have less trivial processes in the sequel). Clearly, the constructed process is stationary ergodic and a B-process. So, we have defined the chain m_0 (and the process ρ_0) up to a parameter δ.

Step 1.1. We begin with the process ρ_0 and the chain m_0 of the previous step. Since the test D is asymptotically correct, we have

$$\mathbf{E}_{\rho_0 \times \rho_0} D_{t_0}\big((x_1, \ldots, x_{t_0}), (y_1, \ldots, y_{t_0})\big) < \varepsilon$$

from some t_0 on, where both samples x_i and y_i are generated by ρ_0 (that is, both samples consist of 1s only). Let k_0 be such an index that the chain m_0 starting from the state 0 with probability 1 does not reach the state $k_0 - 1$ by time t_0 (we can take $k_0 = t_0 + 2$).

Construct two processes ρ_{u1} and ρ_{d1} as follows. They are also based on the Markov chain m_0, but the functions f are different. The function $f_{u1} : \mathbb{N} \to \{0, 1\}$ is defined as follows: $f_{u1}(i) = f_0(i) = 1$ for $i \le k_0$ and $f_{u1}(i) = 0$ for $i > k_0$. The function f_{d1} is identically 1 ($f_{d1}(i) = 1$, $i \in \mathbb{N}$). The processes ρ_{u1} and ρ_{d1} are defined as $\rho_{u1} = f_{u1}(s_t)$ and $\rho_{d1} = f_{d1}(s_t)$ for $t \in \mathbb{N}$. Thus the process ρ_{d1} will again produce only 1s, but the process ρ_{u1} will occasionally produce 0s.

Step 1.2. Being run on two samples generated by the processes ρ_{u1} and ρ_{d1} which both start from the state 0, the test D_n in the first t_0 steps produces many 0s, since on these first k_0 states all the functions f, f_{u1}, and f_{d1} coincide. However, since the processes are different and the test is asymptotically correct (by assumption), the test starts producing 1s, until by a certain time step t_1 almost all answers are 1s. Next we will construct the process ρ_2 by "gluing" together ρ_{u1} and ρ_{d1} and continuing them in such a way that, being run on two samples produced by ρ_2, the test first produces 0s (as if the samples were drawn from ρ_0), and then, with probability close to 1/2, it produces many 1s (as if the samples were from ρ_{u1} and ρ_{d1}) and then again 0s.

The process ρ_2 is the pivotal point of the construction, so we give it in some detail. In step 1.2a we present the construction of the process, and in step 1.2b we show that this process is a B-process by demonstrating that it is equivalent to a (deterministic) function of a Markov chain.

Step 1.2a. Let $t_1 > t_0$ be a time index such that

$$\mathbf{E}_{\rho_{u1} \times \rho_{d1}} D_k\big((x_1, \dots, x_{t_1}), (y_1, \dots, y_{t_1})\big) > 1 - \varepsilon,$$

where the samples x_i and y_i are generated by ρ_{u1} and ρ_{d1}, respectively, (the samples are generated independently; that is, the processes are based on two independent copies of the Markov chain m_0). Let $k_1 > k_0$ be an index such that the chain m starting from the state 0 with probability 1 does not reach the state $k_1 - 1$ by time t_1.

Construct the process ρ_2 as follows (see Fig. 1). It is based on a chain m_2 on which Markov assumption is violated. The transition probabilities on states $0, \dots, k_0$ are the same as for the Markov chain m (from each state return to 0 with probability δ or go to the next state with probability $1 - \delta$).

There are two "special" states, the "switch" S_2 and the "reset" R_2. From the state k_0 the chain passes with probability $1 - \delta$ to the "switch" state S_2. The switch S_2 can itself have two values, *up* and *down*. If S_2 has the value *up*, then from S_2 the chain passes to the state u_{k_0+1} with probability 1, while if $S_2 =$ *down*, the chain goes to d_{k_0+1}, with probability 1. If the chain reaches the state R_2, then the value of S_2 is set to *up* with probability 1/2, and with probability 1/2 it is set to *down*. In other words, the first transition from S_2 is random (either to u_{k_0+1} or to d_{k_0+1} with equal probabilities), and then this decision is remembered until the "reset" state R_2 is visited, whereupon the switch again assumes the values *up* and *down* with equal probabilities.

The rest of the transitions are as follows. From each state u_i, $k_0 \le i \le k_1$ the chain passes to the state 0 with probability δ and to the next state u_{i+1} with probability $1 - \delta$. From the state u_{k_1} the process goes with probability δ to 0 and with probability $1 - \delta$ to the "reset" state R_2. The same with states d_i: for $k_0 < i \le k_1$, the process

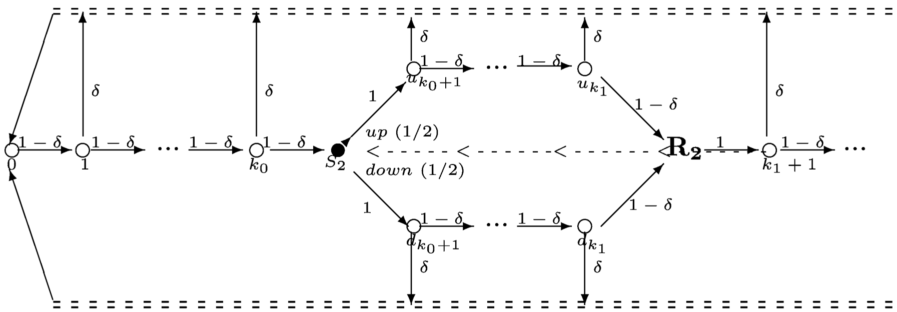

Fig. 1 The processes m_2 and ρ_2. The states are depicted as circles, the arrows symbolize transition probabilities: from every state the process returns to 0 with probability δ or goes to the next state with probability $1-\delta$. From the switch S_2 the process passes to the state indicated by the switch (with probability 1); here it is the state u_{k_0+1}. When the process passes through the reset R_2, the switch S_2 is set to either *up* or *down* with equal probabilities. (Here S_2 is in the position *up*.) The function f_2 is 1 on all states except $u_{k0+1},\dots,u_{k1}$ where it is 0; f_2 applied to the states output by m_2 defines ρ_2

returns to 0 with probability δ or goes to the next state d_{i+1} with probability $1-\delta$, where the next state for d_{k_1} is the "reset" state R_2. From R_2 the process goes with probability 1 to the state k_1+1 where from the chain continues ad infinitum: to the state 0 with probability δ or to the next state k_1+2 with probability $1-\delta$, etc.

The initial distribution on the states is defined as follows. The probabilities of the states $0,\dots,k_0,k_1+1,k_1+2,\dots$ are the same as in the Markov chain m_0, that is, $\delta(1-\delta)^j$ for $j=0,\dots,k_0,k_1+1,k_1+2,\dots$. For the states u_j and d_j, $k_0<j\le k_1$, define their initial probabilities to be $1/2$ of the probability of the corresponding state in the chain m_0, that is, $m_2(u_j)=m_2(d_j)=m_0(j)/2=\delta(1-\delta)^j/2$. Furthermore, if the chain starts in a state u_j, $k_0<j\le k_1$, then the value of the switch S_2 is *up*, and if it starts in the state d_j, then the value of the switch S_2 is *down*, whereas if the chain starts in any other state, then the probability distribution on the values of the switch S_2 is $1/2$ for either *up* or *down*.

The function f_2 is defined as follows: $f_2(i)=1$ for $0\le i\le k_0$ and $i>k_1$ (before the switch and after the reset); $f_2(u_i)=0$ for all i, $k_0<i\le k_1$, and $f_2(d_i)=1$ for all i, $k_0<i\le k_1$. The function f_2 is undefined on S_2 and R_2, and therefore there is no output on these states (we also assume that passing through S_2 and R_2 does not increment time). As before, the process ρ_2 is defined as $\rho_2=f_2(s_t)$, where s_t is the state of m_2 at time t, omitting the states S_2 and R_2. The resulting process is illustrated in Fig. 1.

Step 1.2b. To show that the process ρ_2 is stationary ergodic and a B-process, we will show that it is equivalent to a function of a stationary ergodic Markov chain, whereas all such process are known to be B (e.g., [16]). The construction is as follows (see Fig. 2). This chain has states $k_1+1,\dots$ and also $u_0,\dots,u_{k_0},u_{k_0+1},\dots,u_{k_1}$ and $d_0,\dots,d_{k_0},d_{k_0+1},\dots,d_{k_1}$. From the states u_i, $i=0,\dots,k_1$, the chain passes with probability $1-\delta$ to the next state u_{i+1}, where the next state for u_{k_1} is $k+1$ and with probability δ returns to the state u_0 (and not to the state 0). Transitions for the state $d_0,\dots,d_{k_1-1}$ are defined analogously. Thus the states u_{k_i} correspond to the state *up* of the switch S_2 and the states d_{k_i} to the state *down* of the switch. Transitions for the states $k+1,k+2,\dots$ are defined as follows: with probability $\delta/2$ to the state u_0,

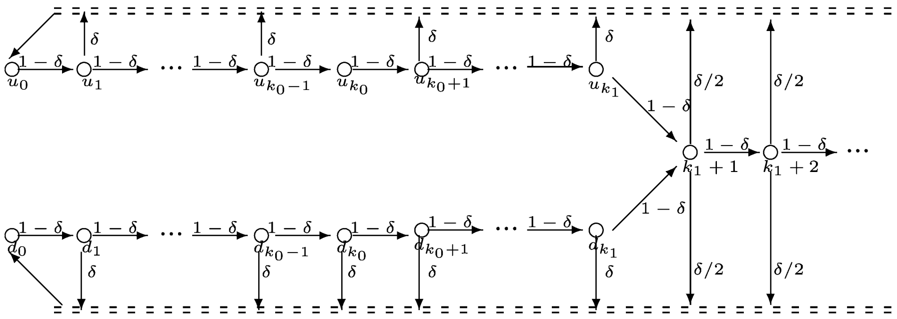

Fig. 2 The process m_2'. The function f_2 is 1 everywhere except the states $u_{k_0+1},\dots,u_{k_1}$, where it is 0

with probability $\delta/2$ to the state d_0, and with probability $1-\delta$ to the next state. Thus, transitions to 0 from the states with indices greater than k_1 correspond to the reset R_2. Clearly, the chain m_2' as defined possesses a unique stationary distribution M_2 over the set of states and $M_2(i)>0$ for every state i. Moreover, this distribution is the same as the initial distribution on the states of the chain m_0, except for the states u_i and d_i, for which we have $m_2'(u_i)=m_2'(d_i)=m_0(i)/2=\delta(1-\delta)^i/2$ for $0\le i\le k_0$. We take this distribution as its initial distribution on the states of m_2'. The resulting process m_2' is stationary ergodic and a B-process, since it is a function of a Markov chain [16]. It is easy to see that if we define the function f_2 on the states of m_2' as 1 on all states except $u_{k_0+1},\dots,u_{k_1}$, then the resulting process is exactly the process ρ_2. Therefore, ρ_2 is stationary ergodic and a B-process.

Step 1.k. As before, we can continue the construction of the processes ρ_{u3} and ρ_{d3} that start with a segment of ρ_2. Let $t_2>t_1$ be a time index such that

$$\mathbf{E}_{\rho_2\times\rho_2}D_{t_2}<\varepsilon,$$

where both samples are generated by ρ_2. Let $k_2>k_1$ be an index such that, when starting from the state 0, the process m_2 with probability 1 does not reach k_2-1 by time t_2 (equivalently: the process m_2' does not reach k_2-1 when starting from either u_0 or d_0). The processes ρ_{u3} and ρ_{d3} are based on the same process m_2 as ρ_2. The functions f_{u3} and f_{d3} coincide with f_2 on all states up to the state k_2 (including the states u_i and d_i, $k_0<i\le k_1$). After k_2 the function f_{u3} outputs 0s, while f_{d3} outputs 1s: $f_{u3}(i)=0$ and $f_{d3}(i)=1$ for $i>k_2$.

Furthermore, we find a time $t_3>t_2$ by which we have $\mathbf{E}_{\rho_{u3}\times\rho_{d3}}D_{t_3}>1-\varepsilon$, where the samples are generated by ρ_{u3} and ρ_{d3}, which is possible since D is consistent. Next, find an index $k_3>k_2$ such that the process m_2 does not reach k_3-1 with probability 1 if the processes ρ_{u3} and ρ_{d3} are used to produce two independent sequences and both start from the state 0. We then construct the process ρ_4 based on a (non-Markovian) process m_4 by "gluing" together ρ_{u3} and ρ_{d3} after the step k_3 with a switch S_4 and a reset R_4 exactly as was done when constructing the process ρ_2. The process m_4 is illustrated in Fig. 3a). The process m_4 can be shown to be equivalent to a Markov chain m_4', which is constructed analogously to the chain m_2' (see Fig. 3b). Thus, the process ρ_4 can be shown to be a B-process.

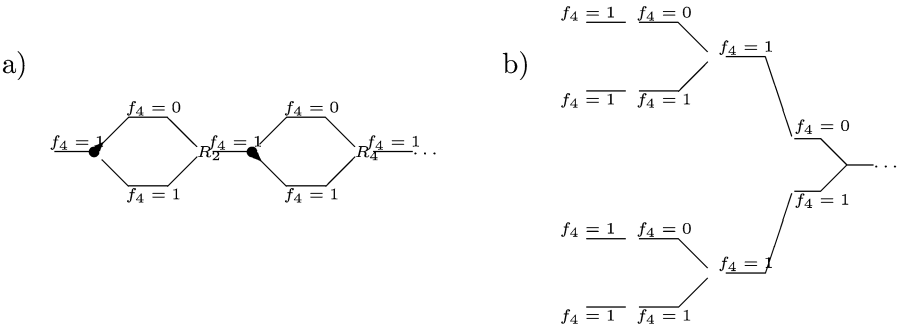

Fig. 3 (**a**) The processes m_4. (**b**) The Markov chain m'_4

Proceeding this way, we can construct the processes ρ_{2j}, ρ_{u2j+1}, and ρ_{d2j+1}, $j \in \mathbb{N}$, choosing the time steps $t_j > t_{j-1}$ so that the expected output of the test approaches 0 by the time t_j being run on two samples produced by ρ_j for even j, and approaches 1 by the time t_j being run on samples produced by ρ_{uj} and ρ_{dj} for odd j:

$$\mathbf{E}_{\rho_{2j}\times\rho_{2j}} D_{t_{2j}} < \varepsilon \tag{4}$$

and

$$\mathbf{E}_{\rho_{u2j+1}\times\rho_{d2j+1}} D_{t_{2j+1}} > (1-\varepsilon). \tag{5}$$

For each j, the number $k_j > k_{j-1}$ is selected in a such a way that the state $k_j - 1$ is not reached (with probability 1) by the time t_j when starting from the state 0. Each of the processes ρ_{2j}, ρ_{u2j+1}, and ρ_{dj2+1}, $j \in \mathbb{N}$, can be shown to be stationary ergodic and a B-process by demonstrating the equivalence to a Markov chain, analogously to Step 1.2. The initial state distribution of each of the processes ρ_t, $t \in \mathbb{N}$, is $M_t(k) = \delta(1-\delta)^k$ and $M_t(u_k) = M_t(d_k) = \delta(1-\delta)^k/2$ for those $k \in \mathbb{N}$ for which the corresponding states are defined.

Step 2. Having defined k_j, $j \in \mathbb{N}$, we can define the process ρ. The construction is given in Step 2a, while in Step 2b we show that ρ is stationary ergodic and a B-process, by showing that it is the limit of the sequence ρ_{2j}, $j \in \mathbb{N}$.

Step 2a. The process ρ can be constructed as follows (see Fig. 4). The construction is based on the (non-Markovian) process m_ρ that has states $0, \dots, k_0$, $k_{2j+1}+1, \dots, k_{2(j+1)}$, $u_{k_{2j}+1}, \dots, u_{k_{2j+1}}$ and $d_{k_{2j}+1}, \dots, d_{k_{2j+1}}$ for $j \in \mathbb{N}$, along with switch states S_{2j} and reset states R_{2j}. Each switch S_{2j} diverts the process to the state $u_{k_{2j}+1}$ if the switch has value *up* and to $d_{k_{2j}+1}$ if it has the value *down*. The reset R_{2j} sets S_{2j} to *up* with probability 1/2 and to *down* also with probability 1/2. From each state that is neither a reset nor a switch, the process goes to the next state with probability $1-\delta$ and returns to the state 0 with probability δ (cf. Step 1k).

The initial distribution M_ρ on the states of m_ρ is defined as follows. For every state i such that $0 \le i \le k_0$ and $k_{2j+1} < i \le k_{2j+2}$, $j = 0, 1, \dots$, define the initial probability of the state i as $M_\rho(i) = \delta(1-\delta)^i$ (the same as in the chain m_0), and for the sets u_j and d_j (for those j for which these sets are defined), let $M_\rho(u_j) =$

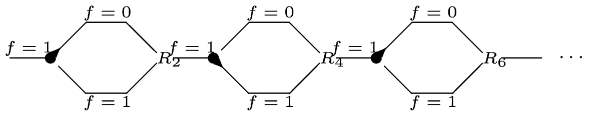

Fig. 4 The processes m_ρ and ρ. The states are on *horizontal lines*. The function f being applied to the states of m_ρ defines the process ρ. Its value is 0 on the states on *the upper lines* (states $u_{k_{2j}+1}, \dots, u_{k_{2j+1}}$, where $k \in \mathbb{N}$) and 1 on the rest of the states

$M_\rho(d_j) := \delta(1-\delta)^i/2$ (that is, 1/2 of the probability of the corresponding state of m_0).

The function f is defined as 1 everywhere except for the states u_j (for all $j \in \mathbb{N}$ for which u_j is defined) on which f takes the value 0. The process ρ is defined at time t as $f(s_t)$, where s_t is the state of m_ρ at time t.

Step 2b. To show that ρ is a B-process, let us first show that it is stationary. Recall Definition 3 of the distributional distance between (arbitrary) process distributions. The set of all stochastic processes, equipped with this distance, is complete, and the set of all stationary processes is its closed subset [3]. Thus, to show that the process ρ is stationary, it suffices to show that $\lim_{j\to\infty} d(\rho_{2j}, \rho) = 0$, since the processes ρ_{2j}, $j \in \mathbb{N}$, are stationary. To do this, it is enough to demonstrate that

$$\lim_{j\to\infty} \bigl|\rho\bigl((x_1, \dots, x_{|B|}) = B\bigr) - \rho_{2j}\bigl((x_1, \dots, x_{|B|}) = B\bigr)\bigr| = 0 \tag{6}$$

for each $B \in X^*$. Since the processes m_ρ and m_{2j} coincide on all states up to k_{2j+1}, we have

$$\begin{aligned}\bigl|\rho(x_n = a) - \rho_{2j}(x_n = a)\bigr| &= \bigl|\rho(x_1 = a) - \rho_{2j}(x_1 = a)\bigr| \\ &\le \sum_{k>k_{2j+1}} M_\rho(k) + \sum_{k>k_{2j+1}} M_{2j}(k)\end{aligned}$$

for all $n \in \mathbb{N}$ and $a \in X$. Moreover, for any tuple $B \in X^*$, we obtain

$$\begin{aligned}&\bigl|\rho\bigl((x_1, \dots, x_{|B|}) = B\bigr) - \rho_{2j}\bigl((x_1, \dots, x_{|B|}) = B\bigr)\bigr| \\ &\quad\le |B|\Biggl(\sum_{k>k_{2j+1}} M_\rho(k) + \sum_{k>k_{2j+1}} M_{2j}(k)\Biggr) \to 0,\end{aligned}$$

where the convergence follows from $k_{2j} \to \infty$. We conclude that (6) holds true, so that $d(\rho, \rho_{2j}) \to 0$, and ρ is stationary.

To show that ρ is a B-process, we will demonstrate that it is the limit of the sequence ρ_{2k}, $k \in \mathbb{N}$, in the $\bar{d}$ distance (which was only defined for stationary processes). Since the set of all B-process is a closed subset of all stationary processes, it will follow that ρ itself is a B-process. (Observe that this way we get the ergodicity of ρ "for free," since the set of all ergodic processes is closed in $\bar{d}$ distance, and all the processes ρ_{2j} are ergodic.) In order to show that $\bar{d}(\rho, \rho_{2k}) \to 0$, we have to find for each j a processes ν_{2j} on pairs $(x_1, y_1), (x_2, y_2), \dots$ such that x_i are distributed according to ρ, y_i are distributed according to ρ_{2j}, and such that $\lim_{j\to\infty} \nu_{2j}(x_1 \neq y_1) = 0$. Construct such a coupling as follows. Consider the chains m_ρ and m_{2j} that start in the same state (with initial distribution being M_ρ) and always take state transitions together, where if the process m_ρ is in the state u_t or d_t, $t \geq k_{2j+1}$ (that is, one of the states that the chain m_{2j} does not have), then the chain m_{2j} is in the state t. The first coordinate of the process ν_{2j} is obtained by applying the function f to the process m_ρ, and the second by applying f_{2j} to the chain m_{2j}. Clearly, the distribution of the first coordinate is ρ, and the distribution of the second is ρ_{2j}. Since the chains start in the same state and always take state transitions together, and since the chains m_ρ and m_{2j} coincide up to the state k_{2j+1}, we have $\nu_{2j}(x_1 \neq y_1) \leq \sum_{k>k_{2j+1}} M_\rho(k) \to 0$. Thus, $\bar{d}(\rho, \rho_{2j}) \to 0$, so that ρ is a B-process.

Step 3. Finally, it remains to show that the expected output of the test D diverges if the test is run on two independent samples produced by ρ.

Recall that for all the chains m_{2j}, m_{u2j+1}, and m_{d2j+1} as well as for the chain m_ρ, the initial probability of the state 0 is δ. By construction, if the process m_ρ starts at the state 0, then up to the time step k_{2j} it behaves exactly as ρ_{2j} that has started at the state 0. In symbols, we have

$$E_{\rho\times\rho}\big(D_{t_{2j}}\big|s_0^x = 0, s_0^y = 0\big) = E_{\rho_{2j}\times\rho_{2j}}\big(D_{t_{2j}}\big|s_0^x = 0, s_0^y = 0\big) \tag{7}$$

for $j \in \mathbb{N}$, where s_0^x and s_0^y denote the initial states of the processes generating the samples x and y, respectively.

We will use the following simple decomposition:

$$\mathbf{E}(D_{t_j}) = \delta^2 \mathbf{E}\big(D_{t_j}\big|s_0^x = 0, s_0^y = 0\big) + \big(1-\delta^2\big)\mathbf{E}\big(D_{t_j}\big|s_0^x \neq 0 \text{ or } s_0^y \neq 0\big). \tag{8}$$

From this, (7), and (4) we have

$$\begin{aligned}\mathbf{E}_{\rho\times\rho}(D_{t_{2j}}) &\leq \delta^2 \mathbf{E}_{\rho\times\rho}\big(D_{t_{2j}}\big|s_0^x = 0, s_0^y = 0\big) + \big(1-\delta^2\big)\\ &= \delta^2 \mathbf{E}_{\rho_{2j}\times\rho_{2j}}\big(D_{t_{2j}}\big|s_0^x = 0, s_0^y = 0\big) + \big(1-\delta^2\big)\\ &\leq \mathbf{E}_{\rho_{2j}\times\rho_{2j}} + \big(1-\delta^2\big) < \varepsilon + \big(1-\delta^2\big).\end{aligned} \tag{9}$$

For odd indices, if the process ρ starts at the state 0, then (from the definition of t_{2j+1}) by the time t_{2j+1} it does not reach the reset R_{2j}; therefore, in this case the value of the switch S_{2j} does not change up to the time t_{2j+1}. Since the definition of m_ρ is symmetric with respect to the values *up* and *down* of each switch, the probability that two samples $x_1, \dots, x_{t_{2j+1}}$ and $y_1, \dots, y_{t_{2j+1}}$ generated independently by (two runs of) the process ρ produce different values of the switch S_{2j} when passing through it for the first time is $1/2$. In other words, with probability $1/2$ two samples generated by ρ starting at the state 0 will look by the time t_{2j+1} as two samples generated by ρ_{u2j+1} and ρ_{d2j+1} that have started at state 0. Thus,

$$E_{\rho\times\rho}\big(D_{t_{2j+1}}\big|s_0^x = 0, s_0^y = 0\big) \geq \frac{1}{2} E_{\rho_{u2j+1}\times\rho_{d2j+1}}\big(D_{t_{2j+1}}\big|s_0^x = 0, s_0^y = 0\big) \tag{10}$$

for $j \in \mathbb{N}$. Using this, (8), and (5), we obtain

$$\begin{aligned}\mathbf{E}_{\rho\times\rho}(D_{t_{2j+1}}) &\geq \delta^2 \mathbf{E}_{\rho\times\rho}\bigl(D_{t_{2j+1}} \big| s_0^x = 0, s_0^y = 0\bigr)\\ &\geq \frac{1}{2}\delta^2 \mathbf{E}_{\rho_{2j+1}\times\rho_{2j+1}}\bigl(D_{t_{2j+1}} \big| s_0^x = 0, s_0^y = 0\bigr)\\ &\geq \frac{1}{2}\bigl(\mathbf{E}_{\rho_{2j+1}\times\rho_{2j+1}}\bigl(D_{t_{2j+1}}\bigr) - \bigl(1-\delta^2\bigr)\bigr) > \frac{1}{2}\bigl(\delta^2 - \varepsilon\bigr). \qquad (11)\end{aligned}$$

Taking δ large and ε small (e.g., $\delta = 0.9$ and $\varepsilon = 0.1$), we can make the bound (9) close to 0 and the bound (11) close to $1/2$, and the expected output of the test will cross these values infinitely often. Therefore, we have shown that the expected output of the test D diverges on two independent runs of the process ρ, contradicting the consistency of D. This contradiction concludes the proof. □

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